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On the Transfer Principle in Fuzzy Theory

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Abstract

We show in this paper that almost all results proved in many papers about fuzzy algebras can be proved uniformly and immediately by using so-called "Transfer Principle".

1 Introduction

There are many papers about fuzzification of algebras so far. Those proved many results which were extentions of some results in the theory of crisp algebras ([1, 2, 3, 4, 5, 9, 10, 11]). But almost all such results are divided into the following four types, which are extensions of those of the crisp theory.

type 0 : A subset A has a property P;

type 1 : If a subset A has a property P, then it has a property Q;

type 2 : Let $f : X \to Y$ be a homomorphism. If a subset B of Y has a property Q, then a subset $f^{-1}(B)$ of X has a property P;

type 3 : Let $f : X \to Y$ be a surjective homomorphism. If a subset A of X has a property P, then a subset $f(A)$ of Y has a property Q.

We now explain these four types with examples.

type 0 : This type is used to define subalgebras, ideals, and so on. Let G be a group. A non-empty subset A of G is called a subgroup when it satisfies the condition

$$x, y \in A \implies xy^{-1} \in A.$$

Then A fuzzy subgroup μ of the group G is defined ([10]) by

$$\mu(0) \geq \mu(x), \quad \mu(xy^{-1}) \geq \mu(x) \wedge \mu(y)$$

type 1 : Let X be a BCK-algebra. A subset A of X is called *ideal* and *positive implicative ideal* when

A : *ideal* $\iff$ (1) $0 \in A$ and (2) $x * y, y \in A$ imply $x \in A$

A : *positive implicative ideal* $\iff$ (1) $0 \in A$ and

(2) $(x*y)*z, y*z \in A$ imply $x*z \in A$

In this case, the following is proved

If A is a positive implicative ideal then it is an ideal.

To extend the result to the fuzzy theory, those definitions are extended in the fuzzy theory as follows: A fuzzy subset μ of X is called *fuzzy ideal, fuzzy positive implicative ideal* when

μ : *fuzzy ideal* $\iff$ (1) $\mu(0) \geq \mu(x)$ and (2) $\mu(x * y) \wedge \mu(y) \leq \mu(x)$

μ : *fuzzy positive implicative ideal* $\iff$ (1) $\mu(0) \geq \mu(x)$ and

(2) $\mu((x * y) * z) \wedge \mu(y * z) \leq$

$\mu(x * z)$

As to these fuzzy ideals, it is proved that

If μ is a fuzzy positive implicative ideal then it is a fuzzy ideal.

type 2 : Let X, Y be BCK-algebras and $f : X \to Y$ be a homomorphism. It is well-known that

If B is an ideal of Y, then $f^{-1}(B)$ is an ideal of X

The result can be extended to the fuzzy theory of BCK-algebras as follows:

If ν is a fuzzy ideal of Y, then $f^{-1}(\nu)$ is a fuzzy ideal of X

type 3 : Let X, Y be BCK-algebras and $f : X \to Y$ be a surjective homomorphism. The next is familiar:

If A is an ideal of X, then $f(A)$ is an ideal of Y

The result is extended to one in the fuzzy theory of BCK-algebras:

If μ is a fuzzy ideal of X, then $f[\mu]$ is a fuzzy ideal of Y

Those results divided into four types are obtained uniformly and immediately by using the transfer principle ([6]). The area the principle can be applied to is restricted to the property which are denoted by a certain formula, but almost all results obtained so far can be represented by a certain formula described below. Thus, our principle is a powerful tool to investigate the fuzzy theory of algebras.

2 Algebras and Terms

To present our theorem accurately, we at first define algebras and terms on the algebras. A structure $(X;\omega_i)_{i\leq k}$ is called an algebra if X is a non-empty set and ω_i is an n_i-ary operation on X. A non-negative interger n_i called an *arity* corresponds to each operation ω_i. A sequence $(n_1, n_2, \cdots, n_k)$ of arities $(n_1 \geq n_2 \geq \cdots \geq n_k)$ is called a *type* of the algebra X. For two algebras $(X;\omega_i)_{i\leq k}$ and $(Y;\xi_j)_{j\leq l}$, they said to be similar if their types are identical.

Next we shall define *terms*. Let $\mathfrak{A}$ be a class of similar algebras and $V = \{x, y, \cdots\}$ be a countable set of variables. A term on an algebra $(X;\omega_i)_{i\leq k} \in \mathfrak{A}$ is defined as follows:

(t1) Each variable $x \in V$ is a term on X;

(t2) If $u_1, \cdots, u_{n_i}$ are terms on X and ω_i is an n_i-ary operation, then $\omega_i(u_1, \cdots, u_{n_i})$ is a term on X.

A term $t(x, \cdots, y)$ is interpreted on an algebra $(X;\omega_i)_{i\leq k}$ as follows:

(Int1) Each variable x is interpreted as an element of X, e.g., $a \in X$.

(Int2) If terms $u_1, \cdots, u_{n_i}$ are interpreted as $a_1, \cdots, a_{n_i} \in X$ respectively, then a term $\omega_i(u_1, \cdots, u_{n_i})$ is done by $\omega_i(a_1, \cdots, a_{n_i}) \in X$.

For the sake of simplicity, we identify a term with the term which is interpreted on some algebra X.

Let $(X;\omega_i)_{i\leq k} \in \mathfrak{A}$. For every subset $A \subseteq X$, we define a formula $\mathcal{P}_A$:

$$\mathcal{P}_A : \forall x \cdots \forall y (t_1(x, \cdots, y) \in A \wedge \cdots \wedge t_n(x, \cdots, y) \in A \to t(x, \cdots, y) \in A),$$

where $t_1(x, \cdots, y), \cdots, t_n(x, \cdots, y)$ and $t(x, \cdots, y)$ are terms on X which are constructed by variables $x, \cdots, y$. We say that a subset A satisfies the formula $\mathcal{P}_A$ when $t_1(a, \cdots, b), \cdots,\ t_n(a, \cdots, b) \in A$ imply $t(a, \cdots, b) \in A$ for every element $a, \cdots, b \in X$. The formula $\mathcal{P}_A$ represents a property of A. In the rest of the paper, we use two statements "property $\mathcal{P}_A$" and "formula $\mathcal{P}_A$" with the same meaning.

Let $(X; \circ, ^{-1}, e)$ be a group. For every non-empty subset $A \subseteq X$, the formula

$$\mathcal{P}_A : \ \forall x \forall y (x \in A \wedge y \in A \to x \circ y^{-1} \in A)$$

means that A is a subgroup of X. We note that non-emptyness is equivalent to a formula

$$\forall x (x \in A \to 0 \in A).$$

Thus we can redefine the concept of subgroup by

$$\forall x (x \in A \to 0 \in A) \text{ and } \forall x \forall y (x \in A \wedge y \in A \to x \circ y^{-1} \in A).$$

For every subset $A \subseteq X$, it is called a $\mathcal{P}$**-set** if it satisfies the formula $\mathcal{P}_A$.

Next we define an algebraic system $\bar{\mathfrak{A}}$ called a *fuzzy theory* of $\mathfrak{A}$. For every algebra $(X;\omega_i)_{i\leq k}$ of $\mathfrak{A}$, a map $\mu : X \to [0, 1]$ is called a *fuzzy subset* of $(X;\omega_i)_{i\leq k}$. The class of all fuzzy subsets of $(X;\omega_i)_{i\leq k} \in \mathfrak{A}$ is said to be a fuzzy theory of $\mathfrak{A}$ and denoted by $\bar{\mathfrak{A}}$.

For every fuzzy subset μ of $X \in \mathfrak{A}$, we define a formula $\mathcal{P}_\mu$ by:

$$\mathcal{P}_\mu : \forall x \cdots \forall y (\mu(t_1(x, \cdots, y)) \wedge \cdots \wedge \mu(t_n(x, \cdots, y)) \leq \mu(t(x, \cdots, y)).$$

Similarly to the case of crisp theory, the formula $\mathcal{P}_\mu$ represents a property of fuzzy subset μ. We say that μ satisfies $\mathcal{P}_\mu$ whenever

$$\mu(t_1(a, \cdots, b)) \wedge \cdots \wedge \mu(t_n(a, \cdots, b)) \leq \mu(t(a, \cdots, b))$$

for all elements $a, \cdots, b \in X$.

In the theory of groups, for example, this means that, for every fuzzy subset μ of X, a *fuzzy subgroup* μ of X is defined by

$$\mu(x) \leq \mu(0) \text{ and } \mu(x) \wedge \mu(y) \leq \mu(x \circ y^{-1}).$$

For every fuzzy subset μ of X, we call μ a ***fuzzy $\mathcal{P}$-set*** if it satisfies the formula $\mathcal{P}_\mu$. For $0 \leq \alpha \leq 1$, we put

$$U(\mu; \alpha) = \{a \in X | \mu(a) \geq \alpha\}.$$

In the following section, we describe a transfer principle accurately and prove important and general theorems using the principle.

3 Transfer Principle

In this section we give an accurate definition of transfer principle and prove the general and fundamental theorems by use of that principle. At first we consider the basic form of a transfer principle. Since this type is basic to develop our theory, we call it **type 0**. In the rest of the paper, let $\mathfrak{A}$ be an arbitrary algebraic system.

Theorem 1 (Transfer Principle (type 0)). *Let $(X; \omega_i) \in \mathfrak{A}$. For every fuzzy subset μ of X,*

$$\mu \textit{ is a fuzzy } \mathcal{P}\textit{-set} \Longleftrightarrow \textit{For all } \alpha \in [0,1], \textit{ if } U(\mu;\alpha) \neq \emptyset \textit{ then } U(\mu;\alpha) \textit{ is a } \mathcal{P}\textit{-set}$$

Proof. ($\Longrightarrow$) Suppose that $U(\mu; \alpha)$ is not a $\mathcal{P}$-set for some $\alpha \in [0, 1]$. There exist elements $a, \cdots, b \in X$ such that

$$t_i(a, \cdots, b) \in U(\mu; \alpha) \ (i \leq n) \text{ but } t(a, \cdots, b) \notin U(\mu; \alpha).$$

Since,

$$\mu(t_i(a, \cdots, b)) \geq \alpha \text{ but } \mu(t(a, \cdots, b)) \not\geq \alpha,$$

we have

$$\mu(t(a, \cdots, b)) \not\geq \bigwedge_i \mu(t_i(a, \cdots, b)).$$

This means that μ is not the fuzzy $\mathcal{P}$-set.

($\Longleftarrow$) Conversely, assume that μ is not a fuzzy $\mathcal{P}$-set. There exist $a, \cdots, b \in X$ such that

$$\mu(t(a, \cdots, b)) \not\geq \bigwedge_i \mu(t_i(a, \cdots, b)).$$

If we take $\alpha = \bigwedge_i \mu(t_i(a, \cdots, b))$, then we have $\alpha \in [0,1]$ $CU(\mu;\alpha) \neq \phi$ and

$$t_i(a, \cdots, b) \in U(\mu;\alpha), \text{ but } t(a, \cdots, b) \notin U(\mu;\alpha).$$

This indicates that $U(\mu;\alpha)$ is not the $\mathcal{P}$-set.

Thus we can prove the theorem completely.

□

Remark : In the fuzzy theory of sets, a similar result called an extentension property is well-known:

$$A = \bigcup_\alpha \alpha U(\mu_A;\alpha)$$

This is a property concering only a set. This does not show the property of operations on the set. In other words, this does not represent the property of a mathematical structure. But our transfer principle does represent the property of the mathematical structure.

The transfer principle means that it is sufficient to ckeck whether a set $U(\mu;\alpha) \neq \emptyset$ satisfies the property $\mathcal{P}$ for all $\alpha \in [0,1]$ if we want to know whether a fuzzy subset μ satisfies a property fuzzy $\mathcal{P}$. Hence we can show the property fuzzy $\mathcal{P}$ of a fuzzy subset μ in the crisp theory of algebras.

We shall prove in the following sections that the transfer principle can be extended to other types (**type 1, type 2, type 3**).

4 Application to type 1

In this section we apply the transfer principle to statements of **type 1** and prove a general theorem of that type. The statement of **type 1** has a form: Let $X \in \mathfrak{A}$ be an arbitrary algebra.

> For every subset A of X, if A has a property $\mathcal{P}$ (or A is a $\mathcal{P}$-set), then it has a property $\mathcal{Q}$ (or it is a $\mathcal{Q}$ set).

We denote the statement simply by

$$\mathfrak{A} \models (X:A):\mathcal{P} \Longrightarrow (X:A):\mathcal{Q}.$$

If the statement $\mathfrak{A} \models (X:A):\mathcal{P} \Longrightarrow (X:A):\mathcal{Q}$ holds for every $X \in \mathfrak{A}$ and $A \subseteq X$, we denote it by

$$\mathfrak{A} \models \mathcal{P} \Longrightarrow \mathcal{Q}.$$

Hence the two formal statements

> For every $X \in \mathfrak{A}$ and $A \subseteq X$, $\mathfrak{A} \models (X:A):\mathcal{P} \Longrightarrow (X:A):\mathcal{Q}$ holds.

and

> $\mathfrak{A} \models \mathcal{P} \Longrightarrow \mathcal{Q}$

have the same meaning.

The statement can be extended to the following in the fuzzy theory of algebras:

> For every fuzzy subset μ of X, if μ has a property $\mathcal{P}_\mu$ (or μ is a fuzzy $\mathcal{P}$-set) , then it has a property $\mathcal{Q}_\mu$ (or it is a fuzzy $\mathcal{Q}$-set).

We also denote the statement simply by

$$\bar{\mathfrak{A}} \models (X:\mu) : \text{fuzzy}\mathcal{P} \Longrightarrow (X:\mu) : \text{fuzzy}\mathcal{Q}.$$

If the statement $\bar{\mathfrak{A}} \models (X:\mu) : \text{fuzzy}\mathcal{P} \Longrightarrow (X:\mu) : \text{fuzzy}\mathcal{Q}$ holds for every $X \in \mathfrak{A}$ and fuzzy subset μ of X, then we denote it by

$$\bar{\mathfrak{A}} \models \text{fuzzy}\mathcal{P} \Longrightarrow \text{fuzzy}\mathcal{Q}.$$

Thus as in the case of above there is no difference between the statement

> For every $X \in \mathfrak{A}$ and μ, we have $\bar{\mathfrak{A}} \models (X:\mu) : \text{fuzzy}\mathcal{P} \Longrightarrow (X:\mu)$: fuzzy$\mathcal{Q}$

and the statement

> $\bar{\mathfrak{A}} \models \text{fuzzy}\mathcal{P} \Longrightarrow \text{fuzzy}\mathcal{Q}$.

We have two examples of the statements of **type 1**.

In the theory of BCK-algebras, if A is an positive implicative ideal then it is an ideal, and in the theory of groups, if A is a normal subgroup then it is a subgroup.

A general theorem concerning to the statements of **type 1** is represented as follows:

Theorem 2. *If* $\mathfrak{A} \models \mathcal{P} \Longrightarrow \mathcal{Q}$ *then* $\bar{\mathfrak{A}} \models \text{fuzzy}\mathcal{P} \Longrightarrow \text{fuzzy}\mathcal{Q}$

Proof. Suppose $\mathfrak{A} \models \mathcal{P} \Longrightarrow \mathcal{Q}$. Let $(X;\omega_i)$ be an arbitrary algebra in $\mathfrak{A}$ and μ a fuzzy subset of X. If μ has a property $\mathcal{P}_\mu$ (i.e., μ is a fuzzy $\mathcal{P}$-set), then it follows from transfer principle that $U(\mu;\alpha) \subseteq X$ is a $\mathcal{P}$-set for all $\alpha \in [0,1]$ such that $U(\mu;\alpha) \neq \emptyset$. Hence $U(\mu;\alpha)$ is a $\mathcal{Q}$-set from assumption. This yields that if $U(\mu;\alpha) \neq \emptyset$ then $U(\mu;\alpha)$ is the $\mathcal{Q}$-set for all $\alpha \in [0,1]$. It follows from transfer principle that μ is the fuzzy $\mathcal{Q}$-set. That is,

$$\bar{\mathfrak{A}} \models \text{fuzzy}\mathcal{P} \Longrightarrow \text{fuzzy}\mathcal{Q}$$

□

Example@Let $\mathfrak{A}$ be a class of groups and $(X;\circ,{}^{-1},e) \in \mathfrak{A}$. For every non-empty subset A of X, we take two formulas $\mathcal{P}_A, \mathcal{Q}_A$ defined by

$\mathcal{P}_A : \forall x \forall y (x \in A \to y \circ x \circ y^{-1} \in A)$;

$\mathcal{Q}_A : \forall x \forall y (x \in A \wedge y \in A \to x \circ y^{-1} \in A)$.

These formulas indicate respectively that

> A is a normal subgroup of X;
>
> A is a subgroup of X.

It is well-known that every normal subgroup is a subgroup. This fact can be represented accurately by

> Let X be an arbitrary group. For every non-empty subset A of X, if A is a normal subgroup of X then it is a subgroup of X.

Thus the fact is denoted by

$$\mathfrak{A} \models \mathcal{P} \Longrightarrow \mathcal{Q}.$$

Hence it follows from the above that

$$\bar{\mathfrak{A}} \models \text{fuzzy}\mathcal{P} \Longrightarrow \text{fuzzy}\mathcal{Q}.$$

The statement represents that

> Let X be an arbitrary group (i.e., $X \in \mathfrak{A}$). For every fuzzy subset μ of X, if μ is a fuzzy normal subgroup then it is a fuzzy subgroup. That is, every normal fuzzy group is a fuzzy group.

It is clear that this statement is an extension of a statement above in the theory of groups to that of the fuzzy theory of groups. Of course any other property of the theory groups can be extended to the fuzzy theory of groups if the property is denoted by the form of this paper.

5 Application to type 2

In this section we apply the transfer principle to the statement of **type 2** which has a follwing form:

> Let $X, Y \in \mathfrak{A}$ be algebras and f be a homomorphism from X to Y. For every subset B of Y, if B is a $\mathcal{Q}$-set then $f^{-1}(B)$ is a $\mathcal{P}$-set.

We denote it formally by

$$\mathfrak{A} \models (Y : B) : \mathcal{Q} \Longrightarrow (X : f^{-1}(B)) : \mathcal{P},$$

If the statement holds for every algebra $X, Y \in \mathfrak{A}$, every homomorphism $f : X \to Y$, and every subset B of Y, then we describe it by

$$\mathfrak{A} \models B : \mathcal{Q} \Longrightarrow f^{-1}(B) : \mathcal{P}.$$

We extend the statement to the case of the fuzzy theory of algebras:

> Let $X, Y \in \mathfrak{A}$ be algebras and f be a homomorphism from X to Y. For every fuzzy subset ν of Y, if ν is a *fuzzy $\mathcal{Q}$-set* then $f^{-1}(\nu)$ is a *fuzzy $\mathcal{P}$-set.*

We denote formally the statement by

$$\bar{\mathfrak{A}} \models (Y:\nu) : \text{fuzzy } \mathcal{Q} \Longrightarrow (X : f^{-1}(\nu)) : \text{fuzzy } \mathcal{P},$$

and if it holds for every algebra $X, Y \in \mathfrak{A}$, every homomorphism $f : X \to Y$, and every fuzzy subset ν of Y, then we do by

$$\bar{\mathfrak{A}} \models \nu : \text{fuzzy } \mathcal{Q} \Longrightarrow f^{-1}(\nu) : \text{fuzzy } \mathcal{P}.$$

The statements of **type 2** have a new concept. It is an inverse image of a fuzzy subset by a homomorphism. To extend the concept to fuzzy theory, we have to define an inverse image of a fuzzy subset by a homomorphism. This is defined as follows: Let X, Y be algebras and f be a homomorphism from X to Y. For every fuzzy subset ν of Y, we define an inverse image $f^{-1}(\nu)$ of ν by

$$f^{-1}(\nu)(x) = \nu(f(x)) \quad (x \in X).$$

where a map $f : X \to Y$ is called a homomorphism when

$$f(\omega_i(a_1, \cdots, a_{n_i})) = \omega_i(f(a_1), \cdots, f(a_{n_i}))$$

for every n_i-ary operation ω_i and for all $a_1, \cdots, a_{n_i} \in X$.

It follows from transfer principle that

Theorem 3. *Let $X, Y \in \mathfrak{A}$ be algebras and $f : X \to Y$ be a homomorphism. Then,*

$$\textit{if } \ \mathfrak{A} \models B : \mathcal{Q} \Longrightarrow f^{-1}(B) : \mathcal{P} \ \ \textit{then } \ \bar{\mathfrak{A}} \models \nu : \text{fuzzy } \mathcal{Q} \Longrightarrow f^{-1}(\nu) : \text{fuzzy } \mathcal{P}.$$

Proof. Let $\mathfrak{A} \models B : \mathcal{Q} \Longrightarrow f^{-1}(B) : \mathcal{P}$, $Y \in \mathfrak{A}$. Suppose that ν is a fuzzy $\mathcal{Q}$-set of Y. It is sufficient from the transfer principle to show that if $U(f^{-1}(\nu); \alpha) \neq \emptyset$ then $U(f^{-1}(\nu); \alpha)$ is a $\mathcal{P}$-set for every $\alpha \in [0, 1]$.

Take any $\alpha \in [0, 1]$ such that $U(f^{-1}(\nu); \alpha) \neq \emptyset$. Since ν is the fuzzy $\mathcal{Q}$-set and

$$U(f^{-1}(\nu); \alpha) = f^{-1}(U(\nu; \alpha)),$$

$U(\nu; \alpha) \neq \emptyset$ and hence $U(\nu; \alpha)$ is a $\mathcal{Q}$-set. From the assumption, $f^{-1}(U(\nu; \alpha)) = U(f^{-1}(\nu); \alpha)$ is the $\mathcal{P}$-set. This means that $f^{-1}(\nu)$ is the fuzzy $\mathcal{P}$-set.

Thus we have $\bar{\mathfrak{A}} \models \nu : \text{fuzzy } \mathcal{Q} \Longrightarrow f^{-1}(\nu) : \text{fuzzy } \mathcal{P}$. □

Example@We explain the power of our theorem by the following example in the theory of BCK-algebras. By BCK-algebra, we mean an algebra $(X; *, 0)$ of type (2,0) satisfying the axioms:

(B1) $((x * y) * (x * z)) * (z * y) = 0$

(B2) $(x * (x * y)) * y = 0$

(B3) $x * x = 0$

(B4) $0 * x = 0$

(B5) $x * y = y * x = 0 \Longrightarrow x = y$

Let $\mathfrak{A}$ be a class of BCK-algebras and $(X; *, 0) \in \mathfrak{A}$. For every non-empty subset A of X, it is called an *ideal* if it satisfies the conditions:

(1) $0 \in A$

(2) $\forall x \forall y(x * y \in A \wedge y \in A \rightarrow x \in A)$

At first glance the condition (1) does not have the form to which the transfer principle can be applied. But we can see easily that the condition is equivalent to the formula:

$$\forall x(x \in X \rightarrow 0 \in A).$$

Thus we can redefine A to be an ideal by

(1)' $\forall x(x \in A \rightarrow 0 \in A)$

(2) $\forall x \forall y(x * y \in A \wedge y \in A \rightarrow x \in A)$.

Clearly we can apply the transfer principle to the formulas. So we define a fuzzy subset μ of X to be a *fuzzy ideal* of X by

(F1) $\forall x(\mu(x) \leq \mu(0))$

(F2) $\forall x \forall y(\mu(x * y) \wedge \mu(y) \leq \mu(x))$.

As an example of **type 2** in the theory of BCK-algebras, we have

Let X, Y be BCK-algebras and $f : X \rightarrow Y$ be a BCK-homomorphism.
If B is an ideal of Y, then $f^{-1}(B)$ is an ideal of X.

It is well-known ([11]) that the result can be extended to the fuzzy theory of BCK-algebras:

Let X, Y be BCK-algebras and $f : X \rightarrow Y$ be a BCK-homomorphism.
If ν is a fuzzy ideal of Y, then $f^{-1}(\mu)$ is a fuzzy ideal of X D

The statements can be represented formally as follows: Let $\mathfrak{A}$ be a class of BCK-algebras, $X, Y \in \mathfrak{A}$, and f be a BCK-homomorphism from X to Y. If we denote that B is an ideal of Y by $\mathcal{Q}_B$ and that $f^{-1}(B)$ is an ideal of X by $\mathcal{P}_{f^{-1}(B)}$, then the original theorem means that

$$\mathfrak{A} \models (Y : B) : \mathcal{Q}_B \Longrightarrow (X : f^{-1}(B)) : \mathcal{P}_{f^{-1}(B)}$$

holds for every $X, Y \in \mathfrak{A}$, BCK-homomorphism $f : X \rightarrow Y$, and $B \subseteq Y$. That is,

$$\mathfrak{A} \models B : \mathcal{Q} \Longrightarrow f^{-1}(B) : \mathcal{P}.$$

Then an extended theorem to the fuzzy theory is represented by

$$\bar{\mathfrak{A}} \models \nu : \text{fuzzy } \mathcal{Q} \Longrightarrow f^{-1}(\nu) : \text{fuzzy } \mathcal{P}.$$

This is exactly the statement of **type 2** and hence immediately from our general theorem.

6 Application to type 3

We define a statement of **type 3** through an example in the theory of BCK-algebras. The statement of **type 3** has a following form:

> Let X, Y be BCK-algebras and $f : X \to Y$ be a surjective homomorphism. If A is an ideal of X, then $f(A)$ is an ideal of Y.

The statement can be extended to the fuzzy theory of BCK-algebras:

> Let X, Y be BCK-algebras and $f : X \to Y$ be a surjective homomorphism. If μ is a *fuzzy ideal* of X, then $f[\mu]$ is a *fuzzy ideal* of Y.

A statement of **type 3** is defined accurately as follows:

> Let X, Y be algebras, $f : X \to Y$ be a homomorphism and A be a subset of X. If A is a $\mathcal{P}$-set, then $f(A)$ is a $\mathcal{Q}$-set.

We formally denote the above by

$$\mathfrak{A} \models (X : A) : \mathcal{P} \Longrightarrow (Y : f(A)) : \mathcal{Q}.$$

If the representation holds for all algebras $X, Y \in \mathfrak{A}$, homomorphism $f : X \to Y$ and $A \subseteq X$, we denote simply

$$\mathfrak{A} \models A : \mathcal{P} \Longrightarrow f(A) : \mathcal{Q}.$$

The statement of **type 3** can be generalized to the fuzzy theory. It has the following representation:

> Let X, Y be algebras, $f : X \to Y$ be a homomorphism and μ be a fuzzy subset of X. If μ is a fuzzy $\mathcal{P}$-set, then $f[\mu]$ is a fuzzy $\mathcal{Q}$-set.

The above can be represented formally as

$$\bar{\mathfrak{A}} \models (X : \mu) : \text{fuzzy } \mathcal{P} \Longrightarrow (Y : f[\mu]) : \text{fuzzy } \mathcal{P}$$

and if the representation holds for all algebras $X, Y \in \mathfrak{A}$, homomorphism $f : X \to Y$ and fuzzy subset of X, then we denote it by

$$\bar{\mathfrak{A}} \models \mu : \text{fuzzy } \mathcal{P} \Longrightarrow f[\mu] : \text{fuzzy } \mathcal{P}.$$

There is also a new concept called an *image* $f[\mu]$ of a fuzzy subset μ by a (surjective) homomorphism f. We have to define $f[\mu]$ for a homomorphism f and a fuzzy subset μ. Let $\mathfrak{A}$ be a class of algebras with similar type, $X, Y \in \mathfrak{A}$ and $f : X \to Y$ be a map from X to Y. For every fuzzy subset μ of X, we define an image $f[\mu]$ of μ as

$$f[\mu](y) = \bigvee_{u \in f^{-1}(y)} \mu(x), \quad y \in Y.$$

If $f^{-1}(y) = \emptyset$, then we put $f[\mu](y) = 0$. We note that the image $f[\mu]$ is also a fuzzy subset of Y. For images of fuzzy subsets we have the fundamental result which plays an important role in the theory of **type 3**.

Lemma 1. *Let $f : X \to Y$ be a surjective homomorphism. For every $\alpha \in [0,1]$, we have*

$$U(f[\mu];\alpha) = \bigcap_{\epsilon>0} f(U(\mu;\alpha-\epsilon))$$

Proof. We can see the result from

$$\begin{aligned} y \in U(f[\mu];\alpha)) &\Longleftrightarrow f[\mu](y) \geq \alpha \\ &\Longleftrightarrow \bigvee_{x\in f^{-1}(y)} \mu(x) \geq \alpha \\ &\Longleftrightarrow \forall\epsilon > 0\ \exists x \in f^{-1}(y) \text{ s.t. } \mu(x) \geq \alpha - \epsilon \\ &\Longleftrightarrow \forall\epsilon > 0\ \exists x \in f^{-1}(y) \text{ s.t. } x \in U(\mu;\alpha-\epsilon) \\ &\Longleftrightarrow \forall\epsilon > 0\ y = f(x) \in f(U(\mu;\alpha-\epsilon)) \\ &\Longleftrightarrow y \in \bigcap_{\epsilon>0} f(U(\mu;\alpha-\epsilon)). \end{aligned}$$

□

We note that it is need to consider the intersection of sets when we use the lemma. In general it arises a problem from considering the intersections of sets. For example, the intersection of non-empty sets is not always a non-empty set. That is, the intersection $\bigcap_\lambda A_\lambda$ of $\{A_\lambda\}_{\lambda\in\Lambda}$ does not always have a property $\mathcal{P}$ even if each set A_λ has the property $\mathcal{P}$. So in order to develop our theory to wider classes of algebras, we need to consider a property whcih is carried over from sets to the intersection of those sets. A property $\mathcal{P}$ is called to have an ***intersection property*** if the intersection $\bigcap_\lambda A_\lambda$ has a property $\mathcal{P}$ for every set A_λ with the property $\mathcal{P}$. This means that if each set A_λ has a property $\mathcal{P}$ then the intersection $A = \bigcap_\lambda A_\lambda$ has the property $\mathcal{P}$, that is,

$$\forall x \cdots \forall y(t_1(x,\cdots,y) \in A_\lambda \wedge \cdots \wedge t_n(x,\cdots,y) \in A_\lambda \to t(x,\cdots,y) \in A_\lambda) \text{ for every } \lambda \in \Lambda \text{ imply}$$
$$\forall x \cdots \forall y(t_1(x,\cdots,y) \in A \wedge \cdots \wedge t_n(x,\cdots,y) \in A \to t(x,\cdots,y) \in A).$$

As examples of those properties (formulas), there are properties of ideals, of subalgebras, of closed sets, and so on.

Using the intersection property we can show the general theorem about the statements of **type 3**.

Theorem 4. *Let $X, Y \in \mathfrak{A}$ and f be a surjective homomorphism from X to Y.*

If $\mathfrak{A} \models A : \mathcal{P} \Longrightarrow f(A) : \mathcal{Q}$ and $\mathcal{Q}$ has an intersection property, then $\bar{\mathfrak{A}} \models \mu :$ fuzzy $\mathcal{P} \Longrightarrow f[\mu] :$ fuzzy $\mathcal{Q}D$

Proof. Suppose that $\mathfrak{A} \models A : \mathcal{P} \Longrightarrow f(A) : \mathcal{Q}$ and $\mathcal{Q}$ has an intersection property. Let $X \in \mathfrak{A}$. We assume that μ is a fuzzy $\mathcal{P}$-set of X. It is sufficient from transfer principle to show that if $U(f[\mu];\alpha) \neq \emptyset$ then $U(f[\mu];\alpha)$ is a $\mathcal{Q}$-set for all $\alpha \in [0,1]$.

Let $\alpha \in [0,1]$ such that $U(f[\mu];\alpha) \neq \emptyset$. It follows from the lemma above that we have

$$U(f[\mu];\alpha) = \bigcap_{\epsilon>0} f(U(\mu;\alpha-\epsilon)).$$

Since $U(f[\mu];\alpha) \neq \emptyset$, we have $\bigcap_{\epsilon>0} f(U(\mu;\alpha-\epsilon)) \neq \emptyset$ and hence $f(U(\mu;\alpha-\epsilon)) \neq \emptyset$ for all $\epsilon > 0$. This implies $U(\mu;\alpha-\epsilon) \neq \emptyset$. From assumption, $U(\mu;\alpha-\epsilon)$ is a $\mathcal{P}$-set for all $\epsilon > 0$ and hence $f(U(\mu;\alpha-\epsilon))$ is the $\mathcal{Q}$-set. Moreover, since $\mathcal{Q}$ has the intersection property, $\bigcap_{\epsilon>0} f(U(\mu;\alpha-\epsilon))$ is also the $\mathcal{Q}$-set, that is, $U(f[\mu];\alpha)$ is the $\mathcal{Q}$-set.

□

As conctrete examples there are the results in the theories of groups and of BCK-algebras. In the theory of groups, it is well-known that

> Let G, G' be groups and $f : G \to G'$ be a surjective homomorphism. If A is a (normal) subgroup of G, then $f(A)$ is a (normal) subgroup of G'.

We also have a result ([9, 10]) which is a generalization to the fuzzy theory of groups.

> Let G, G' be groups and $f : G \to G'$ be a surjective homomorphism. If A is a fuzzy (normal) subgroup of G, then $f(A)$ is a fuzzy (normal) subgroup of G'.

If we consider the properties $\mathcal{P}, \mathcal{Q}$ as (normal) subgroups, then the result is obtained immediately by the theorem above.

We have another example in the theory of BCK-algebras:

> Let X, Y be BCK-algebras and $f : X \to Y$ be a surjective homomorphism. If A is an ideal of X, then $f(A)$ is an ideal of Y.

The result can be extended to the fuzzy theory of BCK-algebras ([4]):

> Let X, Y be BCK-algebras and $f : X \to Y$ be a surjective homomorphism. If μ is a fuzzy ideal of X, then $f[\mu]$ is a fuzzy ideal of Y.

7 Other Properties

Let X be a group. A subset A of X is not always a subgroup of X. In this case we often consider the subgroup $\langle A\rangle$ generated by A, that is, the least subgroup containing A. In this case there is a question whether we can extend such concept to the fuzzy theory of groups. Or more generally, how do we extend the concept to the fuzzy theory of algebras? In this section we think about the question and give a certain solution by use of the transfer principle.

Let $X \in \mathfrak{A}$ be an arbitrary algebra and A be a subset of X. For a formula $\mathcal{P}$ with an intersection property, a $\mathcal{P}$-set $\langle A\rangle$ generated by A is defined as the least $\mathcal{P}$-set which contains A. It can also be represented by

$$\langle A\rangle = \bigcap_\lambda \{B_\lambda \mid A \subseteq B_\lambda,\ B_\lambda : \mathcal{P}\text{-set}\}$$

Hence we can extend the $\mathcal{P}$-set $\langle A\rangle$ generated by A to the fuzzy subset μ of X as follows:

$$\langle \mu\rangle = \bigwedge_\lambda \{\nu_\lambda \mid \mu \le \nu_\lambda,\ \nu_\lambda : \text{fuzzy } \mathcal{P}\text{-set}\},$$

where $\langle\mu\rangle$ is a fuzzy subset of X defined by

$$\langle \mu\rangle(x) = \inf_\lambda \{\nu_\lambda(x) \mid \mu \le \nu_\lambda,\ \nu_\lambda : \text{fuzzy } \mathcal{P}\text{-set }\} \quad (x \in X).$$

It has another representation: For every $\alpha \in [0,1]$,

$$U(\langle\mu\rangle;\alpha) = \bigcap_\lambda \{U(\nu_\lambda;\alpha) \mid \mu \le \nu_\lambda,\ \nu_\lambda : \text{fuzzy } \mathcal{P}\text{-set }\}$$

Using the representation we can get all facts obtained so far by transfer principle and of course we can get new results.

At last we consider a direct product of fuzzy subsets of X. Let μ_i be a fuzzy subset of X_i $(i \in I)$. A map

$$\mu : \Pi_{i\in I} X_i \to [0,1]^I$$

of $\Pi_{i\in I} X_i$ satisfying the condition

$$\mu(a)(j) = \mu_j(a(j)), \quad (a \in \Pi_{i\in I} X_i, \quad j \in I)$$

is called a direct product of μ_i and denoted by $\mu = \Pi_{i\in I}\mu_i$:

As the direct product $\Pi_{i\in I}\mu_i$ of fuzzy subsets μ_i, we have the following.

Lemma 2. *For all $\alpha \in [0,1]^I$,*

$$\Pi_j\, U(\mu_j;\alpha(j)) = U(\Pi_i\mu_i;\alpha)$$

Proof. It is easy to show the lemma from the fact that for every $x \in \Pi_i X_i$

$$\begin{aligned} x \in \Pi_j\, U(\mu_j;\alpha(j)) &\iff x(j) \in U(\mu_j;\alpha(j)) \quad \text{for } \forall j \in I \\ &\iff \mu_j(x(j)) \ge \alpha(j) \quad \text{for } \forall j \in I \\ &\iff (\Pi_i\mu_i)(x) \ge \alpha \\ &\iff x \in U(\Pi_i\mu_i;\alpha) \end{aligned}$$

□

Now we are ready to present a general theorem about the direct product of fuzzy subsets. At first a property $\mathcal{P}$ is said to have a *direct product property* if for every subset X_i with a property $\mathcal{P}$ the direct product $\Pi_i X_i$ also has the property $\mathcal{P}$. Thus, we can get a general theorem about direct product of fuzzy subsets by transfer principle.

Theorem 5. *If $\mathcal{P}$ has a direct product property and $\mathfrak{A} \models A_\lambda : \mathcal{P} \implies \Pi_\lambda A_\lambda : \mathcal{P}$, then $\bar{\mathfrak{A}} \models \mu_\lambda : \text{fuzzy } \mathcal{P} \implies \bar{\mathfrak{A}} \models \Pi_\lambda\mu_\lambda : \text{fuzzy } \mathcal{P}$.*

Proof. It is obvious from the lemma above. □

It follows from the theorem that if a class of crisp algebras is closed under the direct product then a class of extended fuzzy subsets of those algebras is also closed under the direct product.

8 Conclusion

We prove the fundamental and general results that any property about crisp subsets expressed by special formulas can be extended to that of the fuzzy subsets. Thus the direction of the research of fuzzy theory of algebras aims to consider other properties which are not expressed by our formulas. It is also important to investigate the properties **H,S,P** of a class of fuzzy algebras. Of course, these mean that a class of algebras is closed under the operation of *homomorphic images, subalgebras, direct product*, respectively. In this paper we consider the properties **S** and **P**. The rest of ones **H** is very important to consider the quotient algebras. Because to consider the quotient algebras are correspond to do congruences. As to *fuzzy congruences*, their fundamental results are obtained in [7, 8], which are restricted to the case of BCI-algebras and groups but they can be developped to the general case of algebras.

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